A Short Note On

"Taylor & Laurent Series"

(For 4th Semester Physics-Hons. Students)

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Taylor & Maclaurin Series:

We consider a function $f(x)$ defined by power series of the form given by,

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

$$= C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + ----$$

Therefore,

$$f'(x) = C_1 + 2.C_2(x-a) + 3.C_3(x-a)^2 + 4.C_4(x-a)^3 + ----$$
$$f^{(2)}(x) = 2.C_2 + 2.3.C_3(x-a) + 3.4.C_4(x-a)^2 + ------$$
$$f^{(3)}(x) = 2.3.C_3 + 2.3.4.C_4(x-a) + ------$$

and so on

Now if $x = a$;

Then,

$$f(a) = C_0$$
$$f'(a) = C_1$$
$$f^{(2)}(a) = 2.C_2 \quad or, \quad C_2 = \frac{f^{(2)}(a)}{2!}$$
$$f^{(3)}(a) = 2.3.C_3 \quad or, \quad C_3 = \frac{f^{(3)}(a)}{3!}$$

In general we have,

$$f^{(n)}(a) = n!\,C_n \quad or, \quad C_n = \frac{f^{(n)}(a)}{n!}$$

Now suppose that $f(x)$ has a power series expansion at $x = a$ with radius of convergence with R > 0 then the series expansion of $f(x)$ takes the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ----- \qquad (1)$$

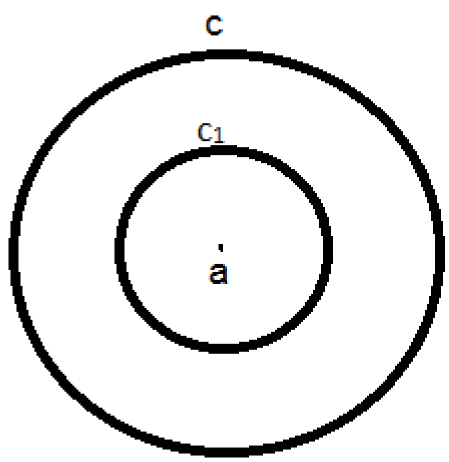
that is the co-efficient C_n in the expansion of $f(x)$ centered at $x = a$ is precisely $C_n = \frac{f^{(n)}(a)}{n!}$. The expansion (1) is called ***Taylor series*** expansion.

If $a = 0$, then the above expansion will take the following form,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + f'(0)\,x + \frac{f^{(2)}(0)}{2!} x^2 + ------ \qquad (2)$$

The expansion (2) is called ***Maclaurin series***.

Connection to Cauchy's integral formula:



The Taylor and Maclaurin's series can be derived by using Cauchy's integral formula. If $f(x)$ is analytic inside a circle `c' with center at `a' then for all x inside `c', we have , $f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ---$

Now consider another circle `$c_1$' center at `a' enclosing x. Then by Cauchy's integral formula,

$$f(x) = \frac{1}{2\Pi i}\oint \frac{f(w)}{w-x}\,dw$$

$$\frac{1}{w-x} = \frac{1}{(w-a)-(x-a)}$$

$$= \frac{1}{w-a}\left\{\frac{1}{1-\frac{(x-a)}{(w-a)}}\right\}$$

$$= \frac{1}{w-a}\left\{1-\frac{(x-a)}{(w-a)}\right\}^{(-1)}$$

$$= \frac{1}{w-a}\left[1+\frac{(x-a)}{(w-a)}+\left\{\frac{(x-a)}{(w-a)}\right\}^{2} + ------ + \left\{\frac{(x-a)}{(w-a)}\right\}^{(n-1)} + ------\right]$$

Therefore,

$$f(x) = \frac{1}{2\Pi i}\oint \frac{f(w)}{w-a}\left[1+\frac{(x-a)}{(w-a)}+\left\{\frac{(x-a)}{(w-a)}\right\}^{2} + ------ + \left\{\frac{(x-a)}{(w-a)}\right\}^{(n-1)} + ------\right] dw$$

$$= \frac{1}{2\Pi i}\oint\frac{f(w)}{w-a}\,dw + \frac{(x-a)}{2\Pi i}\oint\frac{f(w)}{(w-a)^2}\,dw + --- + \frac{(x-a)^{(n-1)}}{2\Pi i}\oint\frac{f(w)}{(w-a)^n}\,dw + --$$

$$= f(a) + f^{'}(a)(x-a) + \frac{f^{(2)}(a)}{2!}\,(x-a)^2 + \frac{f^{(3)}(a)}{3!}\,(x-a)^3$$

The above expansion is known as ***Taylor series*** and if we put a = 0 in this expansion then we shall get ***Maclaurin series.***

Statement:

Let $f(x)$ be analytic inside and on a simple closed curve \`c’. \`a’ , \`a+h’ be two different points inside c, then,

$$f(a+h) = f(a) + f'(a)h + \frac{f^{(2)}}{2!}h^2 + --\mp\frac{f^{(n)}}{n!}h^n$$

Let, $x = a + h\,; h = x - a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

This is called Taylor's theorem and this series is called ***Taylor series***. If we put a=0 in this equation then the resulting series is ,

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + -----.$$

This series is called ***Maclaurin's series***.

<u>Example: Expand sin series by using Taylor / Maclaurin series expansion.</u>

According to Maclaurin series expansion, we know that,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$
$$= f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ------$$

Now if $f(x) = \sin(x)$, then, $f(0) = \sin(0) = 0$

$$f'(x) = \cos(x) \qquad or,\ f'(0) = \cos(0) = 1$$
$$f^{(2)}(x) = -\sin(x) \qquad or,\ f^{(2)}(0) = -\sin(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \qquad or,\ f^{(3)}(0) = -\cos(0) = -1$$
$$f^{(4)}(x) = \sin(x) \qquad or,\ f^{(4)}(0) = \sin(0) = 0$$
$$f^{(5)}(x) = \cos(x) \qquad or,\ f^{(5)}(0) = cos(0) = 1$$

$$f^{(6)}(x) = -\sin(x) \qquad or, \quad f^{(6)}(0) = -\sin(0) = 0$$
$$f^{(7)}(x) = -\cos(x) \qquad or, \quad f^{(7)}(0) = -\cos(0) = -1$$

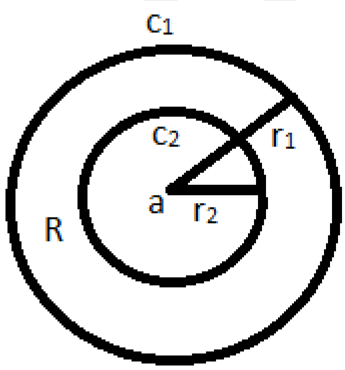
Therefore, $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - - - -$

Home Work:

Derive Cosine and exponential series by using Taylor / Maclaurin's series expansion.

Laurent Series

Statement:



If a function $f(x)$ fails to be analytic at $x = x_0$ we cannot apply Taylor theorem / Taylor series at that point. If $f(x)$ is analytic inside and on the boundary on the ring shaped region and bounded two concentric circle c_1 and c_2 with center at `a' and respective radius r_1 and r_2 then for all x in R

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n + \sum_{n=1}^{\infty} a_{-n} (x-a)^{-n}$$

Where, $a_n = \frac{1}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-a)^{n+1}}\, dw \qquad for\ n = 0,1,2, \dots \dots$

and $a_{-n} = \frac{1}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{-n+1}}\, dw \qquad for\ n = 0,1,2, \dots \dots$

Proof:

By Cauchy's integral formula

$$f(x) = \frac{1}{2\Pi i}\oint \frac{f(w)}{(w-x)} dw$$

$$= \frac{1}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-x)}\, dw - \frac{1}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-x)}\, dw$$

Now $\qquad \frac{1}{w-x} = \frac{1}{(w-a)-(x-a)}$

$$= \frac{1}{w-a}\left\{\frac{1}{1-\frac{(x-a)}{(w-a)}}\right\}$$

$$= \frac{1}{w-a}\left\{1-\frac{(x-a)}{(w-a)}\right\}^{(-1)}$$

$$= \frac{1}{w-a}[1+\frac{(x-a)}{(w-a)}+\left\{\frac{(x-a)}{(w-a)}\right\}^2 + ---+\left\{\frac{(x-a)}{(w-a)}\right\}^{(n-1)} + ---]$$

$$= \frac{1}{(w-a)}+\frac{(x-a)}{(w-a)^2}+\frac{(x-a)^2}{(w-a)^3}+ ---+\frac{(x-a)^n}{(w-a)^n}\frac{1}{(w-a)}+ ---$$

$$\text{Now 1st term} = \frac{1}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-x)}\, dw$$

$$= \frac{1}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-a)}\, dw + \frac{(x-a)}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-a)^2}\, dw$$

$$+ \frac{(x-a)^2}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-a)^3}\, dw + --- + \frac{(x-a)^n}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-a)^{n+1}}\, dw + ---$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + --- + a_n(x-a)^n$$

$$= \sum_{n=0}^{\infty} \mathrm{a_n}(\mathrm{x}-\mathrm{a})^{\mathrm{n}}$$

Where, $a_0 = \; = \frac{1}{2\Pi i}\oint_{c_1} \frac{f(w)}{(w-x)}\, dw$

Again, $\frac{-1}{w-x} = \frac{-1}{(w-a)-(x-a)}$

$$= \frac{1}{x-a}\left\{\frac{1}{1-\frac{(w-a)}{(x-a)}}\right\}$$

$$= \frac{1}{x-a}\left\{1-\frac{(w-a)}{(x-a)}\right\}^{(-1)}$$

$$= \frac{1}{x-a}\left[1+\frac{(w-a)}{(x-a)}+\left\{\frac{(w-a)}{(x-a)}\right\}^2 + --- + \left\{\frac{(w-a)}{(x-a)}\right\}^{(n-1)} + ---\right]$$

$$= \frac{1}{(x-a)} + \frac{(w-a)}{(x-a)^2} + \frac{(w-a)^2}{(x-a)^3} + --- + \frac{(w-a)^{n-1}}{(x-a)^n} + ---$$

Now 2^{nd} term $= \frac{-1}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-x)}\, dw$

$$= \frac{(x-a)^{-1}}{2\Pi i}\oint_{c_2} f(w)\, dw + \frac{(x-a)^{-2}}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{-1}}\, dw + \frac{(x-a)^{-3}}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{-2}}\, dw + ---+ \frac{(x-a)^{-n}}{2\Pi i}\oint_{c_2} \frac{f(w)}{(w-a)^{-n+1}}\, dw + ---$$

$$= \sum_{n=0}^{\infty} a_{-n}(x-a)^{-n}$$

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