A Short Note On

"Singularities and Residue"

(For 4th Semester Physics-Hons. Students)

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<u>Singularities:</u>

In general, a **singularity** is a point at which an equation, surface, etc., blows up or becomes degenerate. **Singularities** are often also called singular points. **Singularities** are extremely important in complex analysis, where they characterize the possible behaviors of analytic functions.

A point at which an analytic function $f(x)$ is not analytic, i.e. at which $f^{'}(x)$ fails to exist, is called a singular point or singularity of the function.

For example the real function $f(x) = \frac{1}{x}$ has a singularity at, $= 0$, where it seems to be “explode” to $\pm\infty$ and is hence not defined.

<u>Different types of singularities:</u>

There are mainly three types of singularities. These are –

1) Removable singularity or Non-essential singularity
2) Poles
3) Essential singularities

1) Removable singularity or Non-essential singularity:

If $\lim_{x \to a} f(x) = finite$

i.e. if a single valued function $f(x)$ is not defined at $x = a$, but $\lim_{x \to a} f(x)$ exist then $x = a$, is called removable or non-essential singularities.

Example:

$$\text{Suppose, } f(x) = \frac{\sin x}{x}$$

$$\text{Now } f(x) \text{ is undefined at } x = 0, \text{ but } \lim_{x \to a} \frac{\sin x}{x} = 1$$

Therefore the function $f(x)$ has a removable or non-essential singularities at $x = 0$.

2) Poles:

$$\text{If } \lim_{x \to a} f(x) = does\ not\ exist$$

Then $x = a$ is called a ''pole''.

Example:

$$\text{Suppose } f(x) = \frac{x}{(x+i)(x^2-9)}$$

The above function has a pole at $x = -i$ and $x = \pm 3$

3) Essential singularities:

If $f(x)$ is single valued, then any singularities which is not a pole or removable singularities, is called essential singularities.

$$\text{If } \lim_{x \to a} f(x) = \infty,$$

then $f(x)$ has an essential singularities at $x = a$.

Example:

$$\text{Suppose } f(x) = e^{\frac{1}{(x-3)}}$$

The above function has an essential singularities at $x = 3$.

Residues and Poles:

Cauchy's theorem tells us that if a function is analytic at all points to and on a simple closed contour C, the value of the integral of function around the closed contour zero. But if the function fails to be analytic at a finite no. of points interior to C, there is a specific no., called a residues which each of those points contributes to the value of the integral.

Calculation of Residues:

If a function has only a finite number of singular points interior to a given closed contour C, they must be isolated.

Let $f(x)$ be a single valued and analytic inside and on a circle C except at the point $x = a$ choosing as the centre of a circle. So the function $f(x)$ has a Lorentz series about $x = a$, given by

$$f(x) = \sum_{n=0}^{\infty} \mathrm{a_n}(x-a)^n + \sum_{n=1}^{\infty} \mathrm{a_{-n}}(x-a)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{a_n}(x-a)^n$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + --- + a_{-1}(x-a)^{-1} + a_{-2}(x-a)^{-2} + ---- \qquad ------- (1)$$

Where, $a_n = \frac{1}{2\Pi i}\oint_C \frac{f(x)}{(x-a)^{n+1}}\, dx \qquad for\ n = 0, \pm 1, \pm 2, ---(2)$

If, $n = -1, a_{-1} = \frac{1}{2\Pi i}\oint_C f(x)\, dx$

Therefore, $\oint_C f(x)\, dx = 2\Pi i a_{-1} \qquad --- (3)$

Here we call a_{-1}be the residue of the function $f(x)$ at $x = a$.

To obtain the residue of a function $f(x)$ at $x = a$, it may appear from Equation-(1) the Lorentz expansion of $f(x)$ about $x = a$ must be obtained. However in the case where $x = a$ is a pole of order k, there is a simple formula for a_{-1} is given by

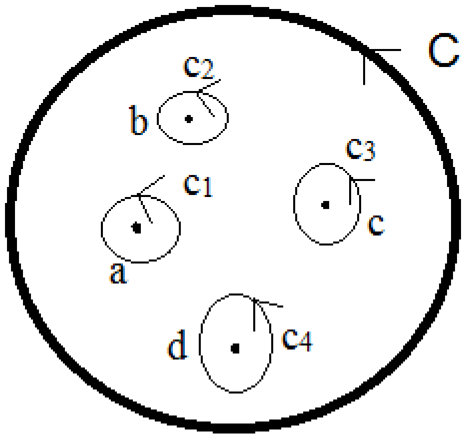
$$a_{-1} = \lim_{x \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}}\{(x-a)^k f(x)\}$$

Residue theorem:

Let C be a positively oriented simple closed contour. If $f(x)$ is analytic inside and on a simple closed curve C except at a finite number of points a, b, c, d, ………….. respectively, then

$$\oint_C f(x)\, dx = 2\Pi i(a_{-1} + b_{-1} + c_{-1} + d_{-1} + ---)$$

With centers at a, b, c, d, ……….. respectively.



Construct a circles c_1, c_2, c_3, c_4, ……….. which lying entirely inside C. This can be done since a, b, c, d, ……….. interior points. So we have

$$\oint_C f(x)\,dx = \oint_{c_1} f(x)\,dx + \oint_{c_2} f(x)\,dx + \oint_{c_3} f(x)\,dx + ---$$

$$= 2\Pi i a_{-1} + 2\Pi i b_{-1} + 2\Pi i c_{-1} + ------$$

$$= 2\Pi i(a_{-1} + b_{-1} + c_{-1} + d_{-1} + ---)$$

Problem – (1)

$$f(x) = \frac{x}{(x-1)(x+1)^2}$$

Find the singular points for the above function and also find out the residue for each singular points.

Singular points, $x = 1$ of poles of order 1

$x = -1$ of poles of order 2

Residue of $f(x)$ at $x = 1$,

$$a_{-1} = \lim_{x \to 1} \frac{1}{0!} \frac{d^0}{dx^0} \left\{ (x-1) \frac{x}{(x-1)(x+1)^2} \right\}$$

$$= \lim_{x \to 1} \frac{x}{(x+1)^2}$$

$$= \frac{1}{4}$$

Residue of $f(x)$ at $x = -1$,

$$a_{-1} = \lim_{x \to -1} \frac{1}{1!} \frac{d}{dx} \left\{ (x+1)^2 \frac{x}{(x-1)(x+1)^2} \right\}$$

$$= \lim_{x \to -1} \frac{d}{dx} \frac{x}{(x-1)}$$

$$= -\frac{1}{4}$$

Another method to find out Residue:

Suppose $f(x) = \frac{F(x)}{G(x)}$

Then $a_{-1} = \left\{ \frac{F(x)}{G'(x)} \right\}_{x = singular\ point}$

Problem – (2)

$$f(x) = \frac{x^2}{x^2 + a^2}$$

Find the singular points for the above function and also find out the residue for each singular points.

Suppose $f(x) = \frac{x^2}{x^2+a^2} = \frac{F(x)}{G(x)}$

Here the function $f(x)$ has a singularity at $x = ia$

Therefore $F(x) = x^2$

$$and \qquad G(x) = x^2 + a^2$$

Therefore $G'(x) = 2x$

$$\text{Then } a_{-1} = \left\{\frac{F(x)}{G'(x)}\right\}_{x=singular\ point}$$

$$= \frac{(ia)^2}{2ia}$$

$$= -\frac{a}{2i}$$