West Bengal State University

B.A./B.Sc./B.Com (Honours, Major, General) Examinations, 2015

PART – III

MATHEMATICS — HONOURS

Paper – V

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

Group – A

[Marks : 70]

Answer Question No. **1** and any *five* from the rest.

1. Answer any *five* questions : $3 \times 5 = 15$

a) Let $I_x = \left(\frac{1}{2}x, \frac{1}{2}(x+1)\right)$, for $x \in (0,1)$. Show that the family $G = \{I_x : x \in (0,1)\}$ is an open cover of $(0,1)$ and no finite subcollection of G can cover $(0,1)$.

b) If $f : [a,b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f(x) \geq 0$, $\forall x \in [a,b]$ and also if $\int_a^b f(x)\,dx = 0$, then prove that $f(x) = 0$, $\forall x \in [a,b]$.

c) Prove that $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a function of bounded variation on $[0,1]$.

d) Find the radius of convergence of the power series $1 + ax + \frac{x^2}{a^2} + a^3x^3 + \frac{x^4}{a^4} + \ldots$; where $|a| < 1$, $a \neq 0$.

e) Examine the convergence of the improper integral $\int_0^{\pi/2} \frac{\cos x}{x^n}\, dx$.

f) Test the uniform convergence of the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = \frac{nx}{1+n^3x^2}$.

g) Obtain Fourier series representation of $f(x) = x$ in $[-\pi, \pi]$.

h) Use Taylor's theorem to express the function $f(x,y) = x^2 + xy + y^2$ in powers of $x-1$ and $y-2$.

i) Evaluate : $\iint_R \sqrt{|x^2 - 2y|}\, dx dy$; where $R = [-2,2\,;\,0,2]$.

2. a) If T is a closed subset of a compact set S in $\mathbb{R}$, then using definition of compact set, prove that T is compact. 3

b) Prove that every closed bounded subset of $\mathbb{R}$ is compact. 5

c) Let $f : S \to \mathbb{R}$ be a continuous function on a compact set $S \subseteq \mathbb{R}$. Prove that $f(S)$ is a compact subset of $\mathbb{R}$. 3

3. a) Prove that a sequence of functions $f_n : S \to \mathbb{R}$; $S \subseteq \mathbb{R}$, $n \in \mathbb{N}$; converges uniformly to a function $f : S \to \mathbb{R}$ if and only if $M_n \to 0$ as $n \to \infty$, where $M_n = \sup_{x \in S} \left| f_n(x) - f(x) \right|$, $n \in \mathbb{N}$. 5

b) If a sequence of bounded functions $f_n : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$, $n \in \mathbb{N}$ converges uniformly to f on S, show that f is also bounded on S. 3

c) Give an example of a sequence of functions $f_n : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$, $n \in \mathbb{N}$ which converges pointwise to $f : S \to \mathbb{R}$ and each f_n is bounded on S but f is not bounded on S. 3

a) State Weierstrass' M-test in connection with uniform convergence of $\sum f_n$ on S, where $f_n : S \to \mathbb{R}$, $n \in \mathbb{N}$, $S \subseteq \mathbb{R}$; and apply it to show that the series of functions $1 + \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} + \frac{e^{-6x}}{6^2-1} + \ldots$ converges uniformly $\forall x \geq 0$. 1 + 3

b) State Abel's test on uniform convergence of a series of functions and apply it to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on [0, 1]. 1 + 3

c) Show that the two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$ have the same radius of convergence. 3

a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable on [a, b] if and only if corresponding to any $\epsilon > 0$, there exists a partition P on [a, b] such that $U(P,f) - L(P,f) < \epsilon$; where symbols have their usual meanings. 4

b) Let $f : [a,b] \to \mathbb{R}$ be bounded on [a, b] and f be continuous on [a, b] except for finite number of points in (a, b). Show that f is Riemann integrable on [a, b]. 4

c) Let $f : [2,5] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \text{ if } x \in Q \cap [2,5] \\ 0 \text{ if } x \in (\mathbb{R} - Q) \cap [2,5] \end{cases}$$

(Q is the set of all rationals). Test Riemann integrability of f on [2, 5]. 3

6. a) If $f:[a,b]\to I\!R$ be continuous on [a, b] and $\phi:[a,b]\to I\!R$ be integrable and maintains the same sign on [a, b], then prove that

$$\int_a^b f(x)\phi(x)\,dx = f(a+\theta(b-a))\int_a^b \phi(x)\,dx, \text{ for } \theta\in[0,1].$$ 4

b) Prove that $1<\int_0^{\pi/2}\sqrt{\sin x}\,dx<\frac{\pi}{2}$. 3

c) State Bonnet's form of second mean value theorem on integral calculus and use it to prove that $\left|\int_a^b \frac{\sin x}{x}dx\right|\le\frac{2}{a};\ 0<a<b<\infty$. 1 + 3

7. a) Let $f:[a,b]\to I\!R$ be continuous on [a, b] and $f'(x)$ exists and bounded on (a, b). Show that f is a function of bounded variation on [a, b]. 3

b) Give an example of a continuous function which is not a function of bounded variation (with justification). 3

c) Obtain the Fourier series expansion of the function $f(x)=x\sin x$ in $[-\pi,\pi]$. Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\dots$. 4 + 1

8 a) Show that the integral $\int_0^2 \frac{\sin x}{x^p}dx$ is convergent if and only if $p<2$. 4

b) Stating the conditions for validity of differentiation under integral sign, prove that $\int_0^{\pi/2}\frac{\log(1+\cos\alpha\cos x)}{\cos x}dx=\frac{1}{8}(\pi^2-4\alpha^2)$. 5

c) Show that $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}=\frac{\pi}{3}$. 2

6. a) If $f:[a,b]\to\mathbb{R}$ be continuous on [a, b] and $\phi:[a,b]\to\mathbb{R}$ be integrable and maintains the same sign on [a, b], then prove that

$$\int_a^b f(x)\phi(x)\,dx = f(a+\theta(b-a))\int_a^b \phi(x)\,dx, \text{ for } \theta\in[0,1].$$ 4

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c) State Bonnet's form of second mean value theorem on integral calculus and use it to prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{2}{a};\ 0<a<b<\infty$. 1 + 3

7. a) Let $f:[a,b]\to\mathbb{R}$ be continuous on [a, b] and $f'(x)$ exists and bounded on (a, b). Show that f is a function of bounded variation on [a, b]. 3

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9. a) Use Lagrange's method to find the points on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ whose distances from the line $3x+y-9=0$ are least and greatest. 4

b) Let $f(x,y)=(1-2xy+x^2)^{-\frac{1}{2}}$. Express $f(1,0)-f(0,1)$ by partial derivatives of f. Hence show that there exists $\theta \in (0,1)$ such that $1-\sqrt{2}=\sqrt{2}(1-3\theta)(1-2\theta+3\theta^2)^{-3/2}$. 3

c) Determine the region of uniform convergence of the series $\sum_{n=1}^{\infty}\frac{2^{n-1}x^{2n-1}}{(4n-3)^2}$. 4

10. a) If $f:[a,b]\rightarrow \mathbb{R}$ be a Lipschitz function on [a, b] then prove that f is a function of bounded variation on [a, b]. Is the converse always true ? Justify your answer. 3 + 1

b) Show that the volume of the solid bounded by the cylinder $x^2+y^2=2ax$ and the paraboloid $y^2+z^2=4ax$ is $\frac{2a^3}{3}(3\pi+8)$. 4

c) Show that $\iint\limits_E \frac{dx\,dy}{\sqrt{(x+y+1)^2-4xy}}=\frac{1}{2}\log_e\left(\frac{16}{e}\right)$, by using the transformation $x=u(1+v)$, $y=v(1+u)$, where E is the triangle with vertices (0,0), (2,0), (2,2). 3

Group – B

[Marks : 15]

Answer any *one* of the following. 1 × 15 = 15

11. a) A function $d:\ \mathbb{R}^2\times\mathbb{R}^2\rightarrow\mathbb{R}$ is defined by $d(\alpha,\beta)=|x_1-y_1|+|x_2-y_2|$, for $\alpha,\beta\in\mathbb{R}^2$, where $\alpha=(x_1,x_2)$, $\beta=(y_1,y_2)$. Show that d is a metric on $\mathbb{R}^2$. 5

b) Given that ($I\!N$, d) is a metric space where $d(m,n)=\frac{|m-n|}{mn}$, for $m,n \in I\!N$. Examine whether ($I\!N$, d) is a complete metric space. 5

c) A is a subset of metric space (X,d). Prove that $x \in \overline{A}$ if and only if there is a sequence $\{x_n\}$ in A converging to x. 5

12. a) If (X,d) is a metric space then prove that (X,d_1) is also a metric space, where $d_1(x,y)=\frac{d(x,y)}{1+d(x,y)}$, for $x,y \in X$. 5

b) In a metric space (X,d), prove that the intersection of a finite number of open sets is open. Does the result hold for any infinite number of open sets ? Justify your answer. 3 + 2

c) Let (X,d) be a complete metric space and $\{F_n\}$ be any sequence of non-empty closed sets such that $F_1 \supseteq F_2 \supseteq \ldots$ in this space with $\lim_{n \to \infty} \delta(F_n)=0$, where $\delta(A)$ denotes the diameter of the set A. Prove that $F=\bigcap_{n=1}^{\infty} F_n$ contains exactly one point in X. 5

Group – C

[Marks : 15]

Answer any *one* of the following. 1 × 15 = 15

13. a) The function $f(z)=u(x,y)+iv(x,y)$ is defined on some neighbourhood of the point $z_0=x_0+iy_0$. Prove that $f(z)$ is continuous at z_0 if and only if both $u(x,y)$ and $v(x,y)$ are continuous at (x_0,y_0). 5

b) Given that ($I\!N$, d) is a metric space where $d(m,n)=\dfrac{|m-n|}{mn}$, for $m,n \in I\!N$. Examine whether ($I\!N$, d) is a complete metric space. 5

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c) Let (X,d) be a complete metric space and $\{F_n\}$ be any sequence of non-empty closed sets such that $F_1 \supseteq F_2 \supseteq \ldots$ in this space with $\underset{n\to\infty}{Lim}\ \delta(F_n)=0$, where $\delta(A)$ denotes the diameter of the set A. Prove that $F=\bigcap_{n=1}^{\infty} F_n$ contains exactly one point in X. 5

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b) Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \frac{(\bar{z})^2}{z}$, $z \neq 0$

$$= 0, z = 0$$

Show that f satisfies Cauchy-Riemann equations at $z = 0$ but the derivative of f fails to exist there. 5

c) If $f(z)$ is differentiable in a region G and $|f(z)|$ is constant in G, then prove that $f(z)$ is constant in G. 5

14. a) Let $f(z) = u(x,y) + iv(x,y)$ be defined in some neighbourhood $N(z_0)$ of z_0 and u, v are both differentiable at (x_0, y_0), where $z_0 = (x_0, y_0)$ and u, v satisfy Cauchy-Riemann equations at (x_0, y_0). Prove that f is differentiable at z_0. 5

b) Show that the function $f(z) = |z|^2$ is differentiable at $z = 0$ but it is not analytic at that point. 2 + 4

c) Use Milne-Thompson method to find an analytic function whose real part is given by $u(x,y) = e^x \cos y$. 4